Announcements

1) Colin Adams (Williams College) will give several talks next week

Tuesday: 3-4, CB 1030
General Audience

Wednesday: 3-4, CB 2046
Research Level

Also a session for math majors, time TBA in CB 2046

Recall: Matrix norm

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$\|T\| = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|_2 = 1}} \|Tx\|_2$$

$$= \max_{\substack{x \in \mathbb{R}^n \\ \|x\|_2 = 1}} \|Tx\|_2$$

Since the Heine-Borel theorem is true in $\mathbb{R}^n$.

Theorem: (uniform continuity)

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be linear.

Then T is uniformly continuous.

Proof: Choose $\varepsilon > 0$. Given $x, y \in \mathbb{R}^n$, we want $\delta > 0$ such that $\|Tx - Ty\|_2 < \varepsilon$ whenever $\|x - y\|_2 < \delta$.

$$\|Tx - Ty\|_2$$

$$\leq \|T\| \, \|x-y\|_2$$

by definition of $\|T\|$

If $Tx = 0$ for all $x \in \mathbb{R}^n$, there is nothing to prove. If $Tx \neq 0$ for some $x \in \mathbb{R}^n$, $\|T\| > 0$. In this case, $\delta = \dfrac{\varepsilon}{\|T\|}$. $\square$

Norm Properties:

Let $A, B : \mathbb{R}^n \to \mathbb{R}^m$

and $C : \mathbb{R}^m \to \mathbb{R}^k$.

Then

1) $\|A+B\| \leq \|A\| + \|B\|$

2) $\|CA\| \leq \|C\| \, \|A\|$

For 2), $x \in \mathbb{R}^n$

$$\|CAx\|_2$$

$$\leq \|C\| \, \|Ax\|_2$$

$$\leq \|C\| \, \|A\| \, \|x\|_2 .$$

$$\Rightarrow \|CA\| \leq \|C\| \cdot \|A\| .$$

Theorem: (invertibility)

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ and suppose T is bijective. Then if T is linear, it admits a linear inverse, denoted by T^{-1}. If $S: \mathbb{R}^n \to \mathbb{R}^m$ linear satisfies

$$\|S - T\| < \frac{1}{\|T^{-1}\|},$$

then S is invertible.

Proof: Consider

$$\| I - T^{-1}S \|$$

where $I : \mathbb{R}^n \to \mathbb{R}^n$, $Ix = x$.

Can write $I = T^{-1}T$. Then

$$\| I - T^{-1}S \|$$
$$= \| T^{-1}T - T^{-1}S \|$$
$$= \| T^{-1}(T - S) \|$$
$$\leq \| T^{-1} \| \, \| T - S \|$$

Then

$$\|I - T^{-1}S\| \le \|T^{-1}\| \, \|T-S\|$$

$$< \|T^{-1}\| \frac{1}{\|T^{-1}\|}$$

$$= 1.$$

Consider, then, the following sum:

$$\boxed{\sum_{n=0}^{\infty} (I - T^{-1}S)^n}$$

good trick!

This defines a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Set $Q = \sum_{n=0}^{\infty} (I - T^{-1}S)^n$

$$(T^{-1}S)Q$$

$$= (T^{-1}S - I + I)Q$$

$$= (T^{-1}S - I)Q + Q$$

$$= -(I - T^{-1}S)Q + Q$$

$$= -\sum_{n=1}^{\infty}(I-T^{-1}S)^n + \sum_{n=0}^{\infty}(I-T^{-1}S)$$

$$= (I-T^{-1}S)^0 = I.$$

This shows that $T^{-1}S$ is invertible, with Q as its inverse. But then $S = T(T^{-1}S)$ is the product of invertible maps, hence invertible. $\square$

Proposition: (derivative of linear transformation)

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $x \in \mathbb{R}^n$.

Then $D(Df(x)) = Df(x)$

proof: Let $y \in \mathbb{R}^n$. Compute

$D(Df(x))(y)$.

This is

$$\lim_{h \to 0} \frac{\|(Df(x))(y+h) - Df(x)y - Ah\|_2}{\|h\|_2}$$

exists for some linear A.

But $(Df(x))(y+h)$

$$= (Df(x))(y) + (Df(x))(h),$$

and hence the numerator becomes

$$\| (Df(x))h - Ah\|_2 .$$

Setting $A = Df(x)$,

we get zero, and

hence the limit

is zero. $\square$

Problem: How do we know derivatives are unique?

Theorem: (uniqueness of derivative)

If $f: \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$, then the derivative is unique.

proof: Suppose $\exists$ linear maps A and B satisfying

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - Ah\|_2}{\|h\|_2}$$

$$= \lim_{h \to 0} \frac{\|f(x+h) - f(x) - Bh\|_2}{\|h\|_2}$$

$$= 0.$$

Choose h, consider

$$\|Ah - Bh\|_2.$$

$$\|Ah - Bh\|_2$$

$$\leq \|Ah - f(x+h) + f(x)\|_2$$

$$+ \|f(x+h) - f(x) - Bh\|_2.$$

Dividing both sides by $\|h\|_2$, we get

$$\frac{\|Ah - Bh\|_2}{\|h\|_2} \leq \frac{\|f(x+h) - f(x) - Ah\|_2}{\|h\|_2} + \frac{\|f(x+h) - f(x) - Bh\|_2}{\|h\|_2}$$

Taking $\lim_{h \to 0}$, we get

$$\lim_{h \to 0} \frac{\|Ah - Bh\|_2}{\|h\|_2} = 0.$$

But $$\frac{\|Ah - Bh\|_2}{\|h\|_2} = \frac{\|(A-B)h\|}{\|h\|_2}$$

$$= \left\|(A-B)\frac{h}{\|h\|_2}\right\|$$

by norm properties and linearity of A and B.

Now write

$$\frac{h}{\|h\|_2} = \frac{th}{\|th\|_2} = \frac{th}{|t|\,\|h\|_2}$$

$$= \frac{th}{t\|h\|_2}$$

if $t > 0$.

As $t \to 0$, $th \to 0$.

So then for a fixed h,

$$\left\| (A-B) \frac{h}{\|h\|_2} \right\|_2$$

$$= \left\| (A-B) \frac{th}{t\|h\|_2} \right\|_2$$

$$= \left\| (A-B) \frac{th}{\|th\|_2} \right\|_2$$

$$\to 0 \text{ as } t \to 0.$$

This implies $(A-B) \frac{h}{\|h\|_2} = 0$

for all $h \in \mathbb{R}^n$, $h \neq 0$.

This implies

$$\|A-B\| = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\|_2 = 1}} \|(A-B)x\|$$

$$= 0, \text{ so } A = B. \quad \square$$

Theorem: (chain rule)

Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$ and suppose $g : \mathbb{R}^m \to \mathbb{R}^k$ is differentiable at $f(x) \in \mathbb{R}^m$.

Then $g \circ f$ is differentiable at x,

$$\boxed{D(g \circ f)(x) = Dg(f(x)) \cdot Df(x)}$$